Properties of pivotal sampling with application to spatial sampling

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Based on joint work with Ronan Le Gleut (Insee)

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Summary

There exists a large number of sampling algorithms, among which systematic sampling is probably the most famous (Madow, 1949 ; Tillé, 2006). It has found applications in a variety of fields.

Systematic sampling enjoys good practical properties, but suffers from a lack of randomness. Some common statistical properties are unlikely to hold, unless explicitly making strong model assumptions (which we try to avoid).

Pivotal sampling appears as a good alternative. While possessing also good practical properties, it introduces more randomness in the sample selection $\Rightarrow$ better statistical properties.

We consider an application for spatial sampling.

Some (short) reminders on sampling

Notations

We are interested in a finite population of statistical units

$$U = \{1, \dots, k, \dots, N\}.$$

Denote by y a variable of interest taking the value y_k for some unit k, and t_y the total.

We note $\pi_k = Pr(k \in S) > 0$ the selection probability of some unit k. The sum $\sum_{k \in U} \pi_k \equiv n$ gives the average sample size.

By using a sampling design matching these inclusion probabilities, the total t_y is unbiasedly estimated by the Horvitz-Thompson (HT) estimator

$$\hat{t}_{y\pi} = \sum_{k \in S} \frac{y_k}{\pi_k} = \sum_{k \in U} \frac{I_k}{\pi_k} y_k, \tag{1}$$

with I_k the sample membership indicator.

Algorithms of sampling

Large number of sampling algorithms matching a prescribed set of inclusion probabilities (see Tillé, 2006). We consider two of them : systematic sampling and pivotal sampling.

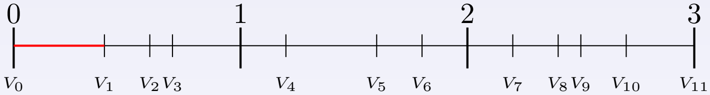
Systematic sampling (Madow, 1949) consists in randomly selecting a first unit, and then performing deterministic jumps to select the remaining units.

Pivotal sampling (Deville and Tillé, 1998 ; Srinivasan, 2001) is based on a principle of duels between units : the units fight, until one of them cumulates a sufficient probability so that a new selection is possible.

Systematic sampling on an example

Population U of size $N = 11$, with $n = 3$ and

$$\pi = (0.4 \;\; 0.2 \;\; 0.1 \;\; 0.5 \;\; 0.4 \;\; 0.2 \;\; 0.4 \;\; 0.2 \;\; 0.1 \;\; 0.2 \;\; 0.3)^\top .$$



We represent the cumulated inclusion probabilities on a segment of length n. Each sub-segment represents one unit.

Systematic sampling on an example

Population U of size $N = 11$, with $n = 3$ and

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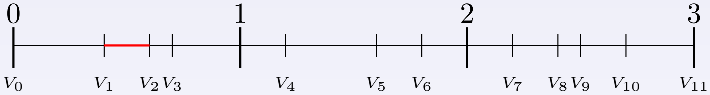

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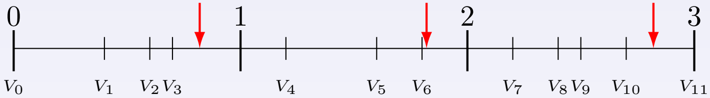

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The sample is obtained through a random start $u \sim U[0,1]$, followed by jumps of length 1.

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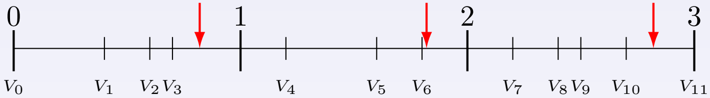


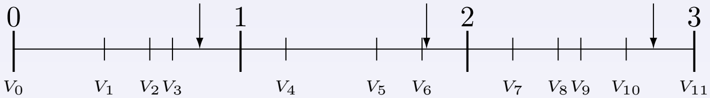
We represent the cumulated inclusion probabilities on a segment of length n. Each sub-segment represents one unit.
The sample is obtained through a random start $u \sim U[0,1]$, followed by jumps of length 1.

$$\begin{aligned} u = 0.82 \in [V_3, V_4] &\quad \Rightarrow \quad \text{unit } 4 \text{ selected,} \\ 1 + u = 1.82 \in [V_6, V_7] &\quad \Rightarrow \quad \text{unit } 7 \text{ selected,} \\ 2 + u = 2.82 \in [V_{10}, V_{11}] &\quad \Rightarrow \quad \text{unit } 11 \text{ selected.} \end{aligned}$$

Systematic sampling on an example (2)

Population U of size $N = 11$, with $n = 3$ and

$$\pi = (0.4 \;\; 0.2 \;\; 0.1 \;\; 0.5 \;\; 0.4 \;\; 0.2 \;\; 0.4 \;\; 0.2 \;\; 0.1 \;\; 0.2 \;\; 0.3)^\top .$$



Very simple method, sequential, matching exactly the π_k's. Extensively used in surveys and in spatial sampling (Thompson, 2002 ; Ripley, 2004).

One unit selected per **microstratum** $\Rightarrow$ stratification effect.

Avoids the selection of neighbouring units $\Rightarrow$ well-spread sample.

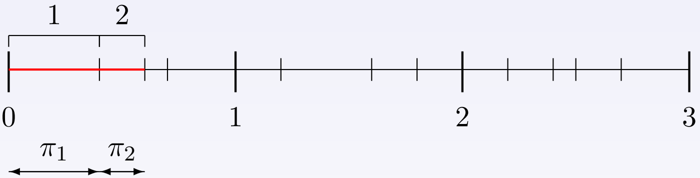
Drawbacks :

- unefficient if the variable of interest exhibits some periodicity,
- very few randomness $\Rightarrow$ limited statistical properties.

Pivotal sampling on an example

Population U of size $N = 11$, with $n = 3$ and

$$\pi \quad = \quad (0.4 \;\; 0.2 \;\; 0.1 \;\; 0.5 \;\; 0.4 \;\; 0.2 \;\; 0.4 \;\; 0.2 \;\; 0.1 \;\; 0.2 \;\; 0.3)^\top .$$

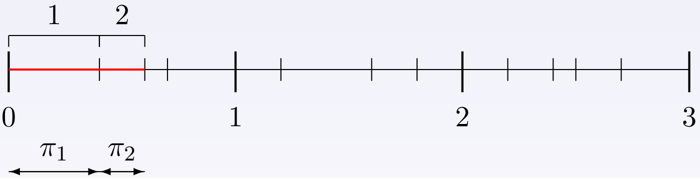


$$(\pi_1, \pi_2) = (0.4, 0.2) = \begin{cases} (0.6, 0) & \text{with proba } 0.4/0.6, \\ (0, 0.6) & \text{with proba } 0.2/0.6 \end{cases}$$

Pivotal sampling on an example

Population U of size $N = 11$, with $n = 3$ and

$$\pi = (\color{red}{0.4}\ \ \color{red}{0.2}\ \ 0.1\ \ 0.5\ \ 0.4\ \ 0.2\ \ 0.4\ \ 0.2\ \ 0.1\ \ 0.2\ \ 0.3)^\top .$$



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If unit 2 survives, we get

$$\pi^{(1)} = (\color{red}{0}\ \ \color{red}{0.6}\ \ 0.1\ \ 0.5\ \ 0.4\ \ 0.2\ \ 0.4\ \ 0.2\ \ 0.1\ \ 0.2\ \ 0.3)^\top .$$

Pivotal sampling on an example (2)

Population U of size $N = 11$, with $n = 3$.

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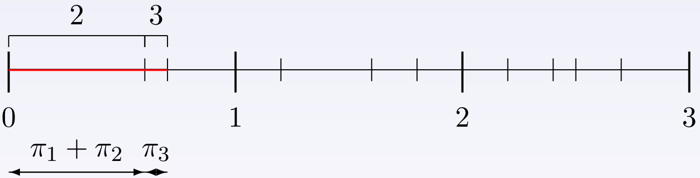

$$(\pi_2^{(1)}, \pi_3^{(1)}) = (0.6, 0.1) = \begin{cases} (0.7, 0) & \text{with proba } 0.6/0.7, \\ (0, 0.7) & \text{with proba } 0.1/0.7 \end{cases}$$

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Population U of size $N = 11$, with $n = 3$.

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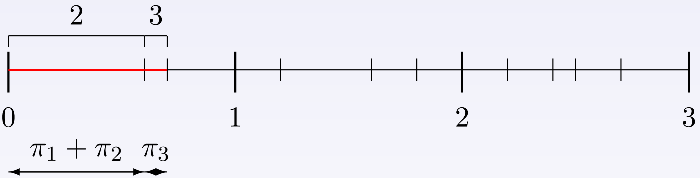

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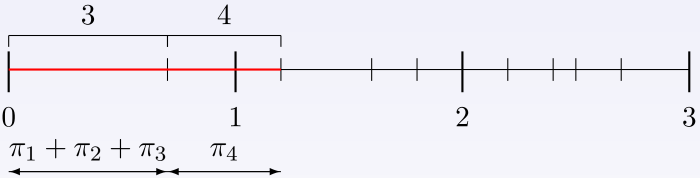
If unit 3 survives, we get

$$\pi^{(2)} = (0 \;\; 0 \;\; 0.7 \;\; 0.5 \;\; 0.4 \;\; 0.2 \;\; 0.4 \;\; 0.2 \;\; 0.1 \;\; 0.2 \;\; 0.3)^\top .$$

Pivotal sampling on an example (3)

Population U of size $N = 11$, with $n = 3$ and

$$\pi^{(3)} = (0 \;\; 0 \;\; \color{red}{0.7} \;\; \color{red}{0.5} \;\; 0.4 \;\; 0.2 \;\; 0.4 \;\; 0.2 \;\; 0.1 \;\; 0.2 \;\; 0.3)^\top .$$



$$(\pi_3^{(2)}, \pi_4^{(2)}) = (0.7, 0.5) = \begin{cases} (1, 0.2) & \text{with proba } 0.5/(2-1.2), \\ (0.2, 1) & \text{with proba } 0.3/(2-1.2) \end{cases}$$

Pivotal sampling on an example (3)

Population U of size $N = 11$, with $n = 3$ and

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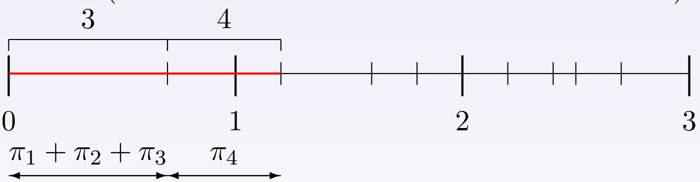

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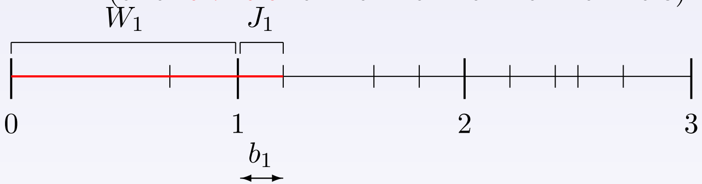
If unit 3 wins, we get

$$\pi^{(3)} = (0 \;\; 0 \;\; \color{red}{1} \;\; \color{red}{0.2} \;\; 0.4 \;\; 0.2 \;\; 0.4 \;\; 0.2 \;\; 0.1 \;\; 0.2 \;\; 0.3)^\top , \ldots$$

Pivotal sampling on an example (4)

Population U of size $N = 11$, with $n = 3$ and

$$\pi^{(3)} = (0 \;\; 0 \;\; \color{red}{0.7} \;\; \color{red}{0.5} \;\; 0.4 \;\; 0.2 \;\; 0.4 \;\; 0.2 \;\; 0.1 \;\; 0.2 \;\; 0.3)^\top .$$



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If unit 3 wins, we get

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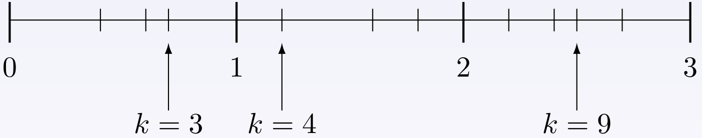
Unit 3 is the first winner (W_1). Unit 4 is the first jumper (J_1).

Pivotal sampling on an example (5)

Population U of size $N = 11$, with $n = 3$ and

$$\pi \;=\; (0.4 \;\; 0.2 \;\; 0.1 \;\; 0.5 \;\; 0.4 \;\; 0.2 \;\; 0.4 \;\; 0.2 \;\; 0.1 \;\; 0.2 \;\; 0.3)^\top .$$



Simple method, sequential, matching exactly the π_k's.
One unit selected per **microstratum** $\Rightarrow$ stratification effect.
Avoids the selection of neighbouring units $\Rightarrow$ well-spread sample.
More randomness $\Rightarrow$ good statistical properties.
Particular case of the cube method (Deville and Tillé, 2004).

Properties of pivotal sampling

Asymptotic set-up and assumptions for CLT

Asymptotic set-up of Fuller (2011) : U belongs to a nested sequence of populations of size $N \to \infty$.

H1 : Non-degenerate : $\exists f_0, f_1$ s.t.

$$0 < f_0 \leq \pi_k \leq f_1 < 1 \text{ for any } k \in U.$$

H2 : Finite moment of order 4 : $\exists C_1$ s.t.

$$\sum_{k \in U} \pi_k \left(\frac{y_k}{\pi_k} - \frac{t_y}{n} \right)^4 \leq C_1 \frac{N^4}{n^3}$$

$$\left[\Leftrightarrow \frac{1}{N} \sum_{k \in U} \left(y_k - \frac{t_y}{N} \right)^4 \leq C_1 \text{ if all } \pi_k' s = \frac{n}{N}. \right]$$

H3 : Non-vanishing variance within microstrata : $\exists C_2 > 0$ s.t.

$$\sum_{i=1}^{n} \sum_{k \in U_i} \alpha_{ik} \left(\frac{y_k}{\pi_k} - \sum_{l \in U_i} \alpha_{il} \frac{y_l}{\pi_l} \right)^2 \geq C_2 \frac{N^2}{n}.$$

Comparison with multinomial sampling

Pivotal sampling design is always more efficient than with-replacement sampling of same size, a.k.a. multinomial sampling (Chauvet, 2017).

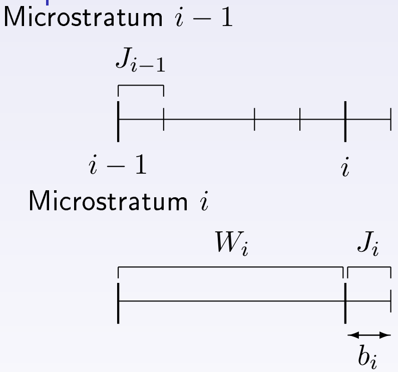
This leads to the variance inequality

$$V_p(\hat{t}_{y\pi}) \quad \leq \quad \sum_{k \in U} \pi_k \left(\frac{y_k}{\pi_k} - \frac{t_y}{n} \right)^2 .$$

Under the assumption H2 (finite 4th moment), the HT-estimator is mean-square consistent for the true total :

$$E_p \left[\left\{ N^{-1} \left(\hat{t}_{y\pi} - t_y \right) \right\}^2 \right] \quad = \quad O(n^{-1}).$$

Martingale decomposition



$$\hat{t}_{y\pi} - t_y = \sum_{i=1}^{n} \underbrace{\left\{ \frac{y_{W_i}}{\pi_{W_i}} + b_i \frac{y_{J_i}}{\pi_{J_i}} - E_{\{\mathcal{F}_{i-1}\}} \left[\frac{y_{W_i}}{\pi_{W_i}} + b_i \frac{y_{J_i}}{\pi_{J_i}} \right] \right\}}_{\text{martingale difference sequence}},$$

$$\frac{\hat{t}_{y\pi} - t_y}{\sqrt{V_p(\hat{t}_{y\pi})}} = \sum_{i=1}^{n} \eta_i.$$

Central-limit theorem

Under Assumptions (H1)-(H3), the estimator $\hat{t}_{y\pi}$ is asymptotically normally distributed (Chauvet and Le Gleut, 2018) :

$$\frac{\hat{t}_{y\pi} - t_y}{\sqrt{V_p(\hat{t}_{y\pi})}} \quad \rightarrow_{\mathcal{L}} \quad \mathcal{N}(0,1).$$

The proof is obtained by checking the sufficient conditions for a martingale central-limit theorem given in Ohlsson (1986) :

$$\begin{aligned} E_p\left(\sum_{i=1}^{n} \eta_i^4\right) & \quad \rightarrow \quad 0, \\ \sum_{i=1}^{n} E_{\{\mathcal{F}_{i-1}\}}(\eta_i^2) & \quad \rightarrow_{Pr} \quad 1. \end{aligned}$$

Problem : design-unbiased variance estimation is not possible.

Conservative variance estimator

To simplify, suppose n even. The difference variance estimator is

$$v_{DIFF}(\hat{t}_{y\pi}) \quad = \quad \sum_{i=1}^{n/2}(1+\delta_i)\left(\frac{y_{W_{2i}}}{\pi_{W_{2i}}} - \frac{y_{W_{2i-1}}}{\pi_{W_{2i-1}}}\right)^2.$$

Omitting the δ_i's, this is the unbiased variance estimator in case of stratification with $n/2$ strata, and multinomial sampling inside.

This estimator is always conservative (Chauvet and Le Gleut, 2018) :

$$E_p\{v_{DIFF}(\hat{t}_{y\pi})\} \quad \geq \quad V_p(\hat{t}_{y\pi}).$$

Noting $\pi_M = \max \pi_k$, we have

$$\delta_i \leq \frac{\pi_M^2(1+\pi_M)}{2(2-\pi_M)} \leq 0.05 \text{ if } \pi_M \leq 0.35.$$

With moderately large inclusion probabilities, we can safely consider :

$$v_{DIFF2}(\hat{t}_{y\pi}) \quad = \quad \sum_{i=1}^{n/2}\left(\frac{y_{W_{2i}}}{\pi_{W_{2i}}} - \frac{y_{W_{2i-1}}}{\pi_{W_{2i-1}}}\right)^2.$$

Spatial sampling

Joint with Ronan Le Gleut (Insee)

Working model

In a context of spatial sampling, first law of geography of Tobler : "Everything is related to everything else, but near things are more related than distant things".

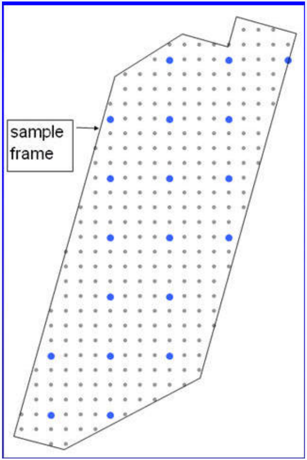
Working model of type (see Grafström and Tillé, 2013) :

$$\begin{array}{rcl} y_k & = & \beta\pi_k + \epsilon_k, \\ E_m(\epsilon_k) = 0 & \text{et} & Cov_m(\epsilon_k, \epsilon_l) = \sigma_k\sigma_l\rho^{d(k,l)}. \end{array}$$

$\Rightarrow$ better to avoid selecting neighbouring units, which carry a similar information.
$\Rightarrow$ better to spread well the sample over space.

More auxiliary information may be available, resulting in more efficient sampling strategies (Grafström and Tillé, 2013).

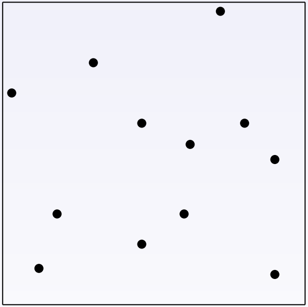
Systematic sampling on a regular grid



- A regular grid is randomly placed on the area under study.
- A sample of points is selected on the grid via systematic sampling.
- The sample is spread over space, but we may face some unexpected periodicity.

Generalized Random Tesselation Sampling (GRTS)

Stevens and Olsen (2004)



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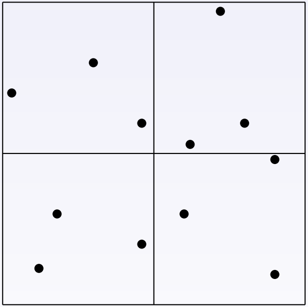

- Tesselation of the area on a regular grid, with "addresses".

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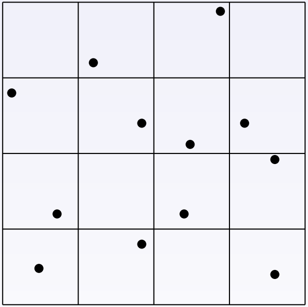

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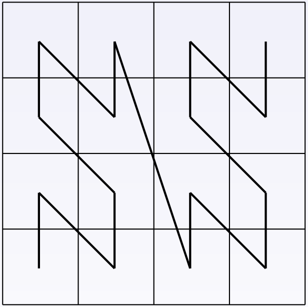

- Tesselation of the area on a regular grid, with "addresses".
- The addresses are ranked on a line.

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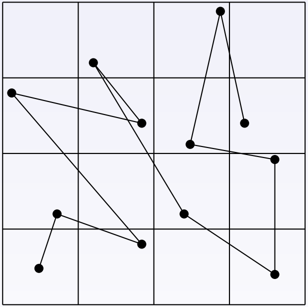

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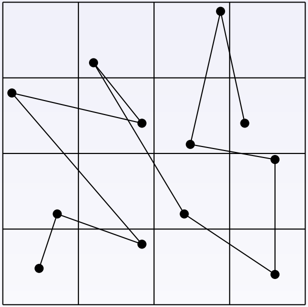

- Tesselation of the area on a regular grid, with "addresses".
- The addresses are ranked on a line.
- Sample selection on the line via systematic sampling after (partial) randomization.

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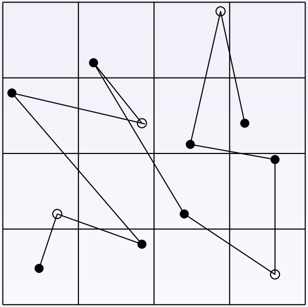

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- Sample selection on the line via systematic sampling after (partial) randomization.

Pivotal Tesselation Method

The GRTS method gives samples well spread over space (Stevens and Olsen, 2004), but with systematic sampling the study of the statistical properties of the HT-estimator is made difficult (and not sure to hold), even with a partial randomization.

We propose to use the tesselation method, but by replacing systematic sampling by pivotal sampling. This leads to the Pivotal Tesselation Method (PTM).

The sample is still well spread over space + HT-estimator consistent and asymptotically normal.

Alternatively, pivotal sampling can be used with any spatial sampling design with some form of ranking on units (e.g., Dickson and Tillé, 2016).

A small simulation study

Example 5 of Grafström et al. (2012). Divide the unit square according to a 20×20 grid $\Rightarrow$ population of $N = 400$ units.
Variable $y_k \equiv$ area within the cell under $f(x1, x2) = 3(x1 + x2) + \sin\{6(x1 + x2)\}$.

Samples of size $n = 16, 32$ or 48 with equal probabilities. Spatial sampling designs :

- pivotal tesselation method (PTM),
- generalized random tesselation sampling (GRTS),
- local pivotal methods (LPM1 and LPM2 ; Grafström et al., 2012).
- pivotal method through Traveling Salesman Problem order (TSP, Dickson and Tillé, 2016).
- simple random sampling (SRS).

Computation of an indicator of spatial balance (Voronoi polygons) + variance associated to each sampling strategy.

Results

Table – Monte Carlo Mean of the spatial balance and Monte Carlo Variance of the Horvitz-Thompson estimator for Population 1

	PTM	GRTS	LPM1	LPM2	TSP	SRS
	$E_{MC}(\Delta)$					
$n = 16$	0.07	0.12	0.08	0.09	0.11	0.33
$n = 32$	0.08	0.11	0.07	0.07	0.10	0.30
$n = 48$	0.09	0.11	0.07	0.07	0.10	0.29
	$V_{MC}(\hat{t}_{y\pi})$ $(\times 100)$					
$n = 16$	1.53	2.49	1.94	1.96	2.65	12.48
$n = 32$	0.39	0.89	0.54	0.57	0.65	6.18
$n = 48$	0.16	0.34	0.26	0.27	0.28	3.91

One step beyond

Sampling in datastreams

Supposing that $|y_k| \leq M$ (strengthening of H2), the martingale writing leads to the exponential inequality

$$Pr\left(\left|\frac{\hat{t}_{y\pi} - t_y}{N}\right| \geq \epsilon\right) \leq 2\exp\left(-\frac{n(C_0)^2\epsilon^2}{M^2}\right) \text{ for any } \epsilon > 0.$$

Enables to find the needed sample size n to ensure a ϵ-δ approximation :

$$Pr\left(\left|\frac{\hat{t}_{y\pi} - t_y}{N}\right| \geq \epsilon\right) \leq \delta,$$

needed when sampling in datastreams (work in progress with Emmanuelle Anceaume, Yann Busnel and Nicolo Rivetti).

Pivotal sampling seems particularly interesting for estimation on the most recent units in the datastream (**sliding window**).

Future work

Pivotal sampling is a particular case of the cube method (Deville and Tillé, 2004), which enables to select balanced samples. A sampling design is balanced on a set x_k of auxiliary variables if

$$\hat{t}_{x\pi}(s) = t_x \quad \text{for all } s \text{ such that} \quad p(s) > 0.$$

A natural question is whether the statistical properties hold in the general case.

Consistency + exponential inequality seems reachable.
The CLT seems more difficult.

Future work

Other spatial sampling methods introduce more complex dependencies in the selection of units :

- local pivotal method (Grafström et al., 2012) : at each step of the pivotal method, the 2 nearest remaining units are treated.
- local cube method (Grafström and Tillé, 2013) : at each step of the cube method, the $p+1$ nearest remaining units are treated.

Similar statistical properties are needed, but the sampling process is informative $\Rightarrow$ much more difficult.

Other fields where pivotal sampling (or other sampling methods) may be of interest ? (Gerber et al., 2018).

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